

A short note on the paper “Remarks on Caristi’s fixed point theorem and Kirk’s problem”

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Abstract: In this paper, we demonstrate that Li’s fixed point theorems are indeed equivalent with the primitive Caristi’s fixed point theorem, Jachymski’s fixed point theorems, Feng and Liu’s fixed point theorems, Khamsi’s fixed point theorems and others.

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Let (X, d) be a metric space. Recall that an operator $T : X \rightarrow X$ is said to be a *Caristi type* mapping [1] provided that the following condition is satisfied

$$\eta(d(x, Tx)) \leq \varphi(x) - \varphi(Tx), \forall x \in X,$$

where $\eta : [0, +\infty) \rightarrow \mathbb{R}$ and $\varphi : X \rightarrow (-\infty, +\infty]$ are functions.

In 1976, Caristi [2] proved the following famous fixed point theorem:

Theorem CA. [2] Let (X, d) be a complete metric space and $\varphi : X \rightarrow \mathbb{R}$ be a lower semicontinuous and bounded below function. Suppose that $T : X \rightarrow X$ is a single-valued mapping satisfying

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \forall x \in X,$$

that is T is a Caristi type mapping with $\eta(t) = t$, $t \in [0, \infty)$. Then T has a fixed point in X .

It is well known that the primitive Caristi’s fixed point theorem is equivalent to Ekeland’s variational principle [3] and to Takahashi’s nonconvex minimization theorem [4, 5]. A number of generalizations in various different directions of these results in metric (or quasi-metric) spaces and more general in topological vector spaces have been investigated by several authors in the past; see [6-19,

22-25] and references therein. But it is worth to mention that some generalizations were real logical equivalent with the original theorems; see e.g. [19, 23-25].

A problem raised by Kirk [20-23, 29, 30] asked whether a Caristi type mapping T for a suitable function η has a fixed point. In fact the original Kirk's question was stated when $\eta(t) = t^p$ for some $p > 1$. Some negative answers to this problem were given; for more detail, one can see [21-23, 25]. Very recently, Li [1] investigated the existence fixed points for Caristi type mappings which partially answered Kirk's problem and improved Caristi's fixed point theorem, Jachymski's fixed point theorems [9, 12], Feng and Liu's fixed point theorems [14], Khamsi's fixed point theorems [23] and others.

The following new fixed point theorems are the main results in [1].

Theorem 1 [1]. Let (X, d) be a complete metric space. Suppose that $\eta : [0, +\infty) \rightarrow \mathbb{R}$ with $\eta(0) = 0$, $\varphi : X \rightarrow \mathbb{R}$ is lower semicontinuous on X , and there exist $x_0 \in X$ and two real numbers $a, \beta \in \mathbb{R}$ such that

$$\varphi(x) \geq ad(x, x_0) + \beta.$$

If one of the following conditions is satisfied:

- (i) $a \geq 0$, η is nonnegative and nondecreasing on $W = \{d(x, y) : x, y \in X\}$, and there exist $c > 0$ and $\varepsilon > 0$ such that

$$\eta(t) \geq ct, \forall t \in \{t \geq 0 : \eta(t) \leq \varepsilon\} \cap W;$$

- (ii) $a < 0$, $\eta(t) + at$ is nonnegative and nondecreasing on W , and there exist $c > 0$ and $\varepsilon > 0$ such that

$$\eta(t) + at \geq ct, \forall t \in \{t \geq 0 : \eta(t) + at \leq \varepsilon\} \cap W.$$

Then each Caristi type mapping $T : X \rightarrow X$ have a fixed point in X .

Theorem 2 [1]. Let (X, d) be a complete metric space. Suppose that $\eta : [0, +\infty) \rightarrow [0, +\infty)$ with $\eta(0) = 0$, $\varphi : X \rightarrow \mathbb{R}$ is lower semicontinuous on X and bounded below on each bounded subset of X , and there exist $x_0 \in X$ and a real number $a \in \mathbb{R}$ such that

$$\liminf_{d(x, x_0) \rightarrow +\infty} \frac{\varphi(x)}{d(x, x_0)} > a.$$

If one of the following conditions is satisfied:

- (i) $a \geq 0$, η is nonnegative and nondecreasing on $[0, +\infty)$, and

$$\liminf_{t \rightarrow 0+} \frac{\eta(t)}{t} > 0;$$

- (ii) $a < 0$, $\eta(t) + at$ is nonnegative and nondecreasing on $[0, +\infty)$, and

$$\liminf_{t \rightarrow 0+} \frac{\eta(t)}{t} \geq -a;$$

Then each Caristi type mapping $T : X \rightarrow X$ have a fixed point in X .

In [1], Li had shown the following.

- Theorem CA \Rightarrow Theorem 1.
- Theorem 1 \Rightarrow Theorem 2.

It is obvious that each of Theorems 1 and 2 implies Theorem CA (the primitive Caristi's fixed point theorem), so we can obtain the following very important result.

Theorem D. Theorem CA, Theorem 1 and Theorem 2 are equivalent.

Remark.

- (a) Li also gave [1, Corollary 1] by using Theorem 2 and he point out that some known extensions of Caristi's fixed point theorem established by Downing and Kirk [6, 7], Jachymski [9, 12], Feng and Liu [14], and Khamsi [23] are special cases of Theorem 2 (for more detail, see [1, Remark 2] and [1, Remark 3]). So, by Theorem D, we know that they are real logical equivalent.
- (b) By a similar argument as Li's results, we can obtain easily that Li's fixed point theorems for other weak distances (w -distances [5, 8, 16] or τ -distances [10, 13] or τ -functions [15, 17, 31] and so on) are equivalent with some weak distance variants of Caristi's fixed point theorem, Ekeland's variational principle and Takahashi's nonconvex minimization; see, e.g., [5, 8, 10, 15-17, 19].

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